PTAssembler's Modified Rectilinear Projections

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Introduction

This document discusses the mathematics behind two new projections that are part of PTAssembler 4.5. These projections were developed to help photographers display wide angle scenes using projections that share some of the important features of standard rectilinear projection, without all of the compromises necessitated by standard cylindrical or spherical projections. Rectilinear projection is important to photographers because it is the only projection that renders horizontal, vertical and diagonal lines as straight. In contrast, cylindrical and spherical projections render some or all of these lines as curved in order to be able to display scenes with fields of view of 180 degrees of more.

The first of the two projections introduced in this document, compressed rectilinear, is similar to rectilinear projection in that it renders both vertical and horizontal lines as straight. It can display scenes with a field of view of up to 180 degrees. The second, recti-perspective, renders both vertical and radial lines as straight. It can display scenes with a field of view of up to 360 degrees.

Terminology

In this document, the following symbols are used:

$$\begin{aligned} \lambda &= yaw = longitude \\ \phi &= pitch = latitude \end{aligned} \tag{1}$$

In cartographers' terms, yaw is referred to as longitude, and pitch is referred to as latitude.

Rectilinear Projection

Rectilinear projection is defined as follows:

$$x = \tan(\lambda) \tag{2}$$

$$y = \tan(\phi)/\cos(\lambda) \tag{3}$$

These equations describe how yaw and pitch (longitude and latitude) values are mapped to corresponding x and y coordinates on a 2 dimensional map or image. As yaw (longitude) increases, values along the x axis become increasingly stretched.

Compressed Rectilinear Projection

PTAssembler's "Compressed Rectilinear" projection is a straightforward modification of rectilinear projection. Its salient feature is that both horizontal and vertical lines remain straight, while the degree to which x axis values are stretched is reduced. It is useful for displaying scenes with a wide field of view where there are prominent horizontal lines, although diagonal lines become curved. It is capable of displaying images with a field of view of up to 180 degrees.

The first step in deriving the compressed rectilinear projection is to scale the x axis, reducing the degree to which the x axis values are stretched as yaw moves away from zero. This can be done in a variety of ways, but a simple approach is to as follows:

$$x' = \alpha \tan(\lambda / \alpha) \qquad (4)$$

x' is the x value for compressed rectilinear projection. Higher values for α reduce the stretching associated with large values for yaw. As α increases, x' becomes increasingly close to a linear function of λ, similar to cylindrical, equirectangular and other projections.

The y axis can also be scaled in a number of ways, but the simplest approach is to use simple linear scale factor as follows:

$$\begin{aligned} y' &= \beta y \\ &= \beta \tan(\phi) / \cos(\lambda) \end{aligned} \qquad (5)$$

When β is larger than 1, the y axis appears stretched. Values less than 1 give the appearance of compressing the y axis.

Recti-Perspective Projection

PTAssembler's recti-perspective projection is a modification of compressed rectilinear projection. The x axis value is the same as compressed rectilinear projection:

$$x' = \alpha \tan(\lambda / \alpha) \tag{6}$$

Vertical lines are still rendered as straight using recti-perspective projection, but not horizontal lines. Instead, radial lines (*i.e.* lines that pass through the center of the image) are rendered as straight, and it is capable of displaying images of up to 360 degrees.

In order for recti-perspective projection to render radial lines as straight, all y axis values must be scaled using a constant multiple of the factor used to scale corresponding values along the x axis. This scale factor is defined as γ :

$$\gamma = \beta(x'/x) \tag{7}$$

where β is a factor that is used to compress or expand the y axis.

The value for the y axis becomes:

$$
\begin{aligned}
y' &= \gamma y \\
&= \beta (x'/x) y \\
&= \beta (\alpha \tan(\lambda/\alpha)/\tan(\lambda)) * (\tan(\phi)/\cos(\lambda))
\end{aligned} \tag{8}
$$

This formulation is undefined when yaw reaches 90 degrees. However, noting that $\tan(\lambda)\cos(\lambda)$ is equal to $\sin(\lambda)$ the formula can be rewritten as:

$$y' = (\beta \alpha \tan(\lambda/\alpha) \tan(\phi))/\sin(\lambda) \tag{9}$$

When the values of α and β equal 1, this formula simplifies to standard rectilinear projection. When α is larger than 1, images with a horizontal field of view of more than 180 degrees can be displayed. Once α reaches a value of 2, scenes of up to 360 degrees can be displayed. However, the larger the value for α , the more horizontal lines above or below the horizon appear to curve.

Note that this formulation is undefined when yaw is equal to zero. However, as yaw approaches zero, both $\cos(\lambda)$ and x'/x approach one (see formula 8). So, when yaw equals zero, the y value can be calculated as:

$$y' = \beta \tan(\phi) \tag{10}$$

Conclusion

The two new projections discussed in this document are useful to photographers interested in displaying scenes with a wide field of view, but unwilling to live with the compromises imposed by standard cylindrical or spherical projections. Both of these projections are featured in PTAssembler 4.5. For more information, please contact maxlyons@tawbaware.com

Links

PTAssembler: http://www.tawbaware.com/ptasmblr.htm

Projection discussion and example images: http://www.tawbaware.com/projections.htm